Lecture 14:

Recall:

Def: Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The **algebraic multiplicity** of λ, denoted $\mu_T(\lambda)$ or $\mu_A(\lambda)$ is the multiplicity of λ as a zero of $f(t)$, i.e. the largest positive integer k s.t. $(t-\lambda)^k \mid f(t)$.

(e.g. $f(t) = (t-1)^3 (t-4)^4 (t-5)^7$

Alg. mult. of $\lambda = 1$ is 3

$\lambda = 4$ is 4

$\lambda = 5$ is 7)

Example: $\cdot$ 1 is eigenvalue of $I_V : \underset{\beta}{V} \to \underset{\beta}{V}$

with $\mu_{I_V}(1) = \dim(V)$

$$f(t) = \det\Big(\underset{\substack{\| \\ I_n}}{[I_V]_\beta} - t I_n \Big) = \begin{pmatrix} 1-t & & & \\ & 1-t & & \\ & & \ddots & \\ & & & 1-t \end{pmatrix} = (1-t)^n$$

Prop: Let T be a linear operator on a finite-dim vector space V and let λ be an eigenvalue of T with algebraic multiplicity $\mu_T(\lambda)$. Then:

$$1 \leq \dim(E_\lambda) \leq \mu_T(\lambda)$$

We call $\gamma_T(\lambda) \overset{\text{def}}{=} \dim(E_\lambda)$ the geometric multiplicity of λ.

Proof: Choose an ordered basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ for E_λ and extend it to an ordered basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_p, \vec{v}_{p+1}, \ldots, \vec{v}_n\}$ for V.

$p = \dim(E_\lambda)$

Then: $$[T]_\beta = \begin{pmatrix} | & & | & \\ [\underbrace{T(\vec{v}_1)}_{=\lambda\vec{v}_1}]_\beta & \cdots & [\underbrace{T(\vec{v}_p)}_{=\lambda\vec{v}_p}]_\beta & \cdots \\ | & & | & \end{pmatrix} = \begin{pmatrix} \lambda & 0 & & 0 & \\ 0 & \lambda & & 0 & \\ 0 & 0 & \cdots & \vdots & \\ \vdots & \vdots & & \lambda & \\ & & & 0 & \\ 0 & 0 & & \vdots & \\ & & & 0 & \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \lambda I_p & B \\ \hline O & C \end{array}\right)$$

$$\Rightarrow \quad f_T(t) = \det\left(\begin{array}{c|c} (\lambda - t) I_p & B \\ \hline 0 & C - t I_{n-p} \end{array}\right)$$

$$= \det\left((\lambda - t) I_p\right) \underline{\det\left(C - t I_{n-p}\right)}$$

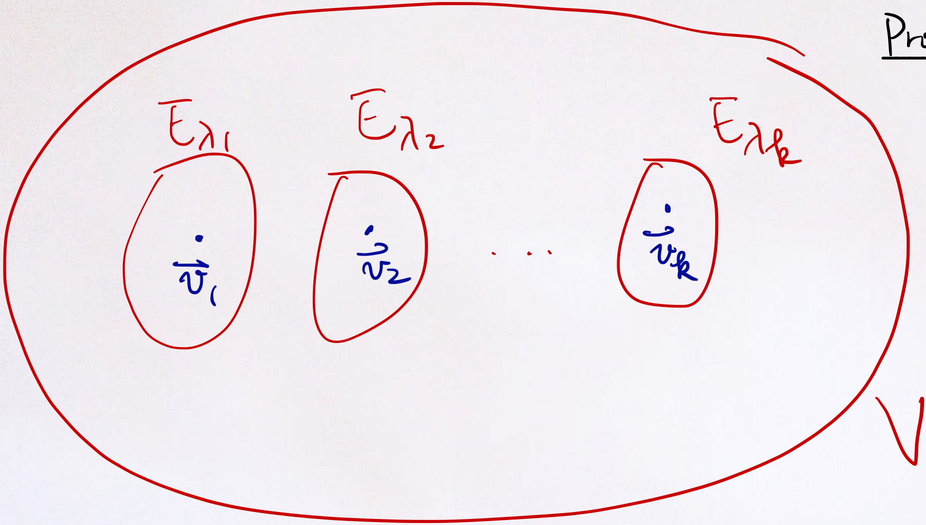
$$= (\lambda - t)^p \det\left(C - t I_{n-p}\right)$$

$$\therefore \quad (\lambda - t)^p \mid f_T(t)$$

$$\therefore \quad \mu_T(\lambda) \geq p = \gamma_T(\lambda)$$

Lemma: Let T be a linear operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ distinct eigenvalues of T. For each $i = 1, 2, \ldots, k$, let $\vec{v}_i \in E_{\lambda_i}$.

If $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k = \vec{0}$, then $\vec{v}_i = \vec{0}$ for all i.



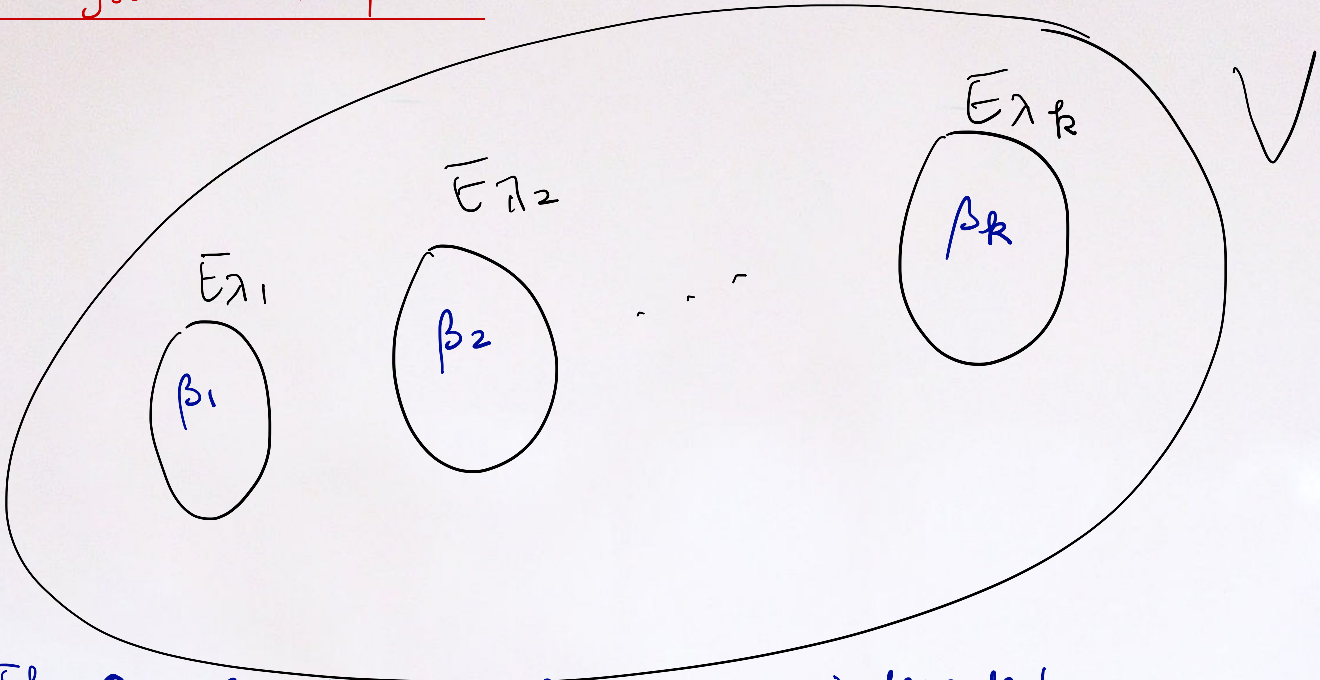
Proof: If not, say $\vec{v}_1, \ldots, \vec{v}_s \neq \vec{0}$

then:

$$\underset{\neq \vec{0}}{\vec{v}_1} + \underset{\neq \vec{0}}{\vec{v}_2} + \ldots + \underset{\neq \vec{0}}{\vec{v}_s} = \vec{0}$$

It contradicts to our previous proposition that $\vec{v}_1, \ldots, \vec{v}_s$ must be lin. independent.

Our goal is to prove:



Then: ① $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$ is linear independent.

② If $|\beta_1| + |\beta_2| + \dots + |\beta_k| = \dim(V)$, then β is a basis of eigenvectors.

Proposition: Let T be a linear operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, 2, \ldots, k$, let $S_i \subset E_{\lambda_i}$ be a finite linearly independent subset. Then:

$$S = S_1 \cup S_2 \cup \cdots \cup S_k \text{ is a linearly independent}$$

subset of V.

Proof: Write $S_i = \{\vec{v}_{i1}, \vec{v}_{i2}, \ldots, \vec{v}_{in_i}\}$ for $i = 1, 2, \ldots, k$.

Suppose $\exists\, a_{ij} \in F$ for $1 \le j \le n_i$ and $1 \le i \le k$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} \vec{v}_{ij} = \vec{0}$$

$\overset{\text{def}}{=}$ $w_i \in E_{\lambda_i}$

Then: $\underset{\in E_{\lambda_1}}{w_1} + \underset{\in E_{\lambda_2}}{w_2} + \cdots + \underset{\in E_{\lambda_k}}{w_k} = \vec{0} \implies w_i = \sum_{j=1}^{n_i} a_{ij} \vec{v}_{ij} = \vec{0}$ for all i.

Then: $a_{ij} = 0$ for all i and j

(for S_i are lin. independent for all i.

$\therefore S_1 \cup S_2 \cup \dots \cup S_k$ is linearly independent.

Theorem: Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial $f_T(t)$ splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T.

Then: (a) T is diagonalizable iff: $\mu_T(\lambda_i) = \gamma_T(\lambda_i)$ for $i = 1, 2, \ldots, k$

(b) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then: $\beta := \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors.

(so that $[T]_\beta$ is a diagonal matrix)

Pf: Next time!!